Fourier and Wavelet Transformations in Geometric Algebra

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First we introduce basic Geometric Algebra (GA) multivector functions. For these functions we define the coordinate independent vector differential and the vector derivative. This includes explicit examples and basic differential geometric calculus rules.

Then we motivate and define the GA Fourier Transformation (GA FT) for the GA of real Euclidean 3-space and give examples of its application. We give an overview of its most important properties (higher dimensional geometric generalizations of scalar complex Fourier Transformation properties). We show how to generalize the GA FT to higher dimensions and explain what role characteristic GA non-commutativity plays. Known applications include uncertainty, LSI filters (smoothing, edge detection), signal analysis, image processing, fast (multi)vector pattern matching, visual flow analysis, sampling, (multi)vector field analysis. GA FTs can be discretised and fast GA FT algorithms are available.

Next we introduce several types of so called Quaternion Fourier Transformations (QFT). We explain their mutual relations, genuine 2D phase properties, their geometric transformation properties, discrete versions and fast numerical implementations. Applications include, partial differential systems, color image processing, filtering, etc. Geometric Algebra relationships enable wide ranging higher dimensional generalizations. As an example we generalize to a Spacetime Fourier Transform, which naturally leads to multivector wave packet analysis in physics, and directional uncertainty, now with additional geometric insight.

While the GA FT is global, we introduce in the last part the local GA wavelet concept in the GA of real Euclidean 2-space and Euclidean 3-space, using two dimensional translations, dilations and rotations combined in the similitude group SIM(2), or SIM(3), respectively. Multivector mother wavelet functions need to fullfil the admissibility condition, which includes an admissibility constant with scalar and vector parts. We define the invertible GA wavelet transformation and discuss its main properties. An explicit example is the GA Gabor multivector wavelet.