Rational trigonometry and Vector Trigonometry

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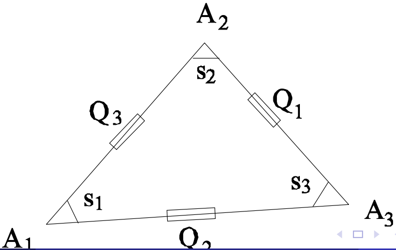
Rational Trigonometry: Quadrance and spread

A **point** A is an ordered pair $[x, y]$ of numbers. The **quadrance** $Q(A_1, A_2)$ between points $A_1 \equiv [x_1, y_1]$ and $A_2 \equiv [x_2, y_2]$ is the number

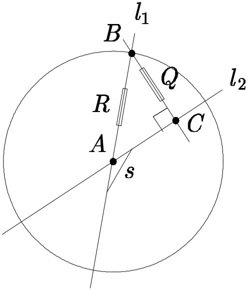
$$Q(A_1, A_2) \equiv (x_2 - x_1)^2 + (y_2 - y_1)^2 = d^2(A_1, A_2)$$

A **line** l is an ordered proportion $\langle a : b : c \rangle$, representing the equation $ax + by + c = 0$. The **spread** $s(l_1, l_2)$ between lines $l_1 \equiv \langle a_1 : b_1 : c_1 \rangle$ and $l_2 \equiv \langle a_2 : b_2 : c_2 \rangle$ is the number

$$s(l_1, l_2) \equiv \frac{(a_1 b_2 - a_2 b_1)^2}{(a_1^2 + b_1^2)(a_2^2 + b_2^2)} = \sin^2 \theta$$



Angle versus spread



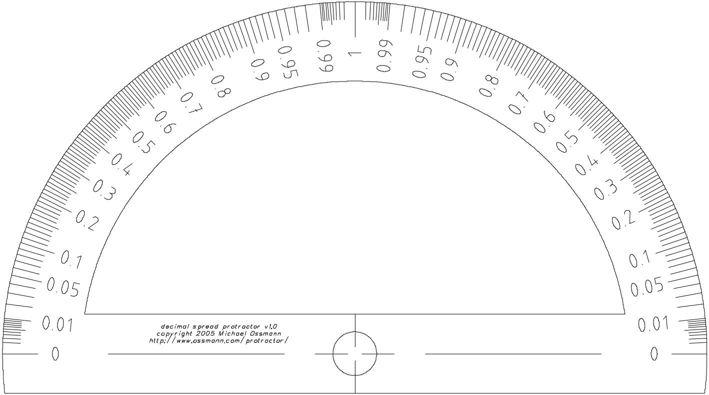
Geometric interpretation of spread:

$$s(l_1, l_2) \equiv \frac{Q(B, C)}{Q(A, B)} = \frac{Q}{R}.$$

Special values and a Spread Protractor

You may check that the spread corresponding to 30° or 150° or 210° or 330° is $s = 1/4$, the spread corresponding to 45° or 135° etc. is $s = 1/2$, and the spread corresponding to 60° or 120° etc. is $3/4$, while the spread corresponding to 90° is 1.

The following *spread protractor* was created by M. Ossmann.



Five main laws of Rational Trigonometry

Pythagoras' theorem The lines A_1A_3 and A_2A_3 are perpendicular precisely when

$$Q_1 + Q_2 = Q_3.$$

Triple quad formula The three points A_1, A_2 and A_3 are collinear precisely when

$$\left(Q_1 + Q_2 + Q_3\right)^2 = 2\left(Q_1^2 + Q_2^2 + Q_3^2\right).$$

Spread law For any triangle $\overline{A_1A_2A_3}$

$$\frac{s_1}{Q_1} = \frac{s_2}{Q_2} = \frac{s_3}{Q_3}.$$

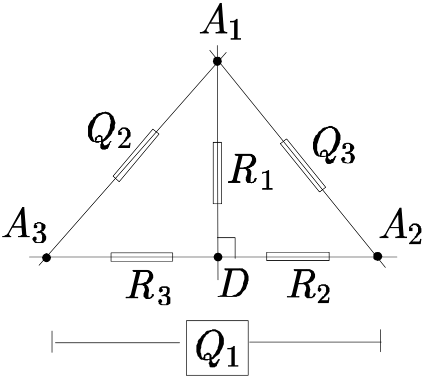
Cross law For any triangle $\overline{A_1A_2A_3}$

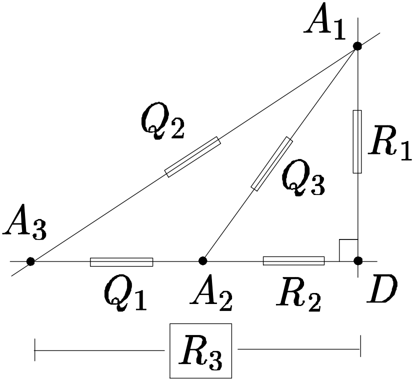
$$\left(Q_1 + Q_2 - Q_3\right)^2 = 4Q_1Q_2\left(1 - s_3\right)$$

Triple spread formula For any triangle $\overline{A_1A_2A_3}$

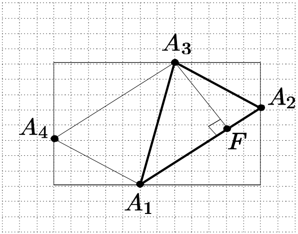
$$\left(s_1 + s_2 + s_3\right)^2 = 2\left(s_1^2 + s_2^2 + s_3^2\right) + 4s_1s_2s_3.$$

Proofs





Heron's formula



The area of the triangle $\overline{A_1A_2A_3}$ is one half of the area of the associated parallelogram $\overline{A_1A_2A_3A_4}$.

The latter area may be calculated by removing from the circumscribed 12×8 rectangle four triangles, which can be combined to form two rectangles, one 5×3 and the other 7×5. The area of $\overline{A_1A_2A_3}$ is thus 23.

Heron's formula *If* $s \equiv (d_1 + d_2 + d_3)/2$ *is the semi-perimeter of a triangle, then its area is*

$$\text{area} = \sqrt{s(s-d_1)(s-d_2)(s-d_3)}.$$

Heron's formula calculation

In the previous example:

$$d_1 = \sqrt{34} \qquad d_2 = \sqrt{68} \qquad d_3 = \sqrt{74}.$$

The semi-perimeter s, defined to be one half of the sum of the side lengths, is then

$$s = \frac{\sqrt{34} + \sqrt{68} + \sqrt{74}}{2} \approx 11.339\,744\,206\,6\ldots.$$

Using the usual Heron's formula, a computation with the calculator shows that

$$\text{area} = \sqrt{s\left(s - \sqrt{34}\right)\left(s - \sqrt{68}\right)\left(s - \sqrt{74}\right)} \approx 23.000\,000.$$

Archimedes theorem

Theorem (Archimedes)

The area of a triangle $\overline{A_1 A_2 A_3}$ with quadrances Q_1, Q_2 and Q_3 is given by

$$16 \,\mathrm{area}^2 = (Q_1 + Q_2 + Q_3)^2 - 2\left(Q_1^2 + Q_2^2 + Q_3^2\right).$$

In our example the triangle has quadrances $34, 68$ and 74, each obtained by Pythagoras' theorem. So Archimedes' theorem states that

$$16 \,\mathrm{area}^2 = (34 + 68 + 74)^2 - 2\left(34^2 + 68^2 + 74^2\right) = 8464$$

and this gives an area of 23. In rational trigonometry, the quantity

$$\mathcal{A} = (Q_1 + Q_2 + Q_3)^2 - 2\left(Q_1^2 + Q_2^2 + Q_3^2\right)$$

is the **quadrea** of the triangle, and turns out to be the single most important number associated to a triangle.

Extended spread law

The Cross law

$$\left(Q_1 + Q_2 - Q_3\right)^2 = 4Q_1 Q_2 \left(1 - s_3\right)$$

may be rewritten as

$$\mathcal{A} = \left(Q_1 + Q_2 + Q_3\right)^2 - 2\left(Q_1^2 + Q_2^2 + Q_3^2\right) = 4Q_1 Q_2 s_3$$

so we get an **Extended Spread law**:

$$\frac{s_1}{Q_1} = \frac{s_2}{Q_2} = \frac{s_3}{Q_3} = \frac{\mathcal{A}}{4Q_1 Q_2 Q_3}.$$

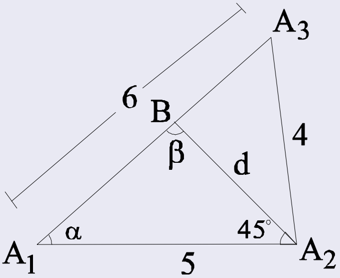
So to calculate spreads, find the quadrea $\mathcal{A}$ first, then

$$s_1 = \frac{\mathcal{A}}{4Q_2 Q_3} \qquad \text{etc.}$$

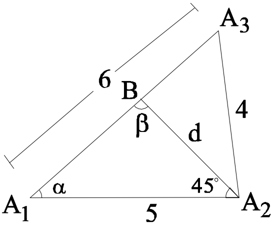
Example problem

Problem

The triangle $\overline{A_1A_2A_3}$ *has side lengths* $|A_1A_2| = 5$, $|A_2A_3| = 4$ *and* $|A_3A_1| = 6$. *The point* B *is on the line* A_1A_3 *with the angle between* A_1A_2 *and* A_2B *equal to* $45°$. *What is the length* $d \equiv |A_2B|$?



Classical solution



$$4^2 = 5^2 + 6^2 - 2 \times 5 \times 6 \times \cos\alpha$$

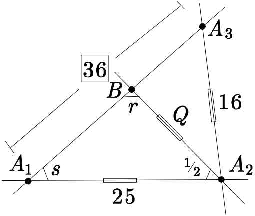
$$\alpha = \arccos\frac{3}{4} \approx 41.4096^\circ.$$

$$\beta \approx 180^\circ - 45^\circ - 41.4096^\circ \approx 93.5904^\circ.$$

$$\frac{\sin\alpha}{d} = \frac{\sin\beta}{5}$$

$$d \approx \frac{5\sin 41.4096^\circ}{\sin 93.5904^\circ} \approx 3.313\,691\,689\,613.$$

Rational solution



Cross law in $\overline{A_1 A_2 A_3}$:

$$(25 + 36 - 16)^2 = 4 \times 25 \times 36 \times (1 - s) \qquad \text{so that} \qquad s = 7/16.$$

Triple spread formula in $\overline{A_1 A_2 B}$:

$$\left(\frac{7}{16} + \frac{1}{2} + r \right)^2 = 2 \left(\frac{49}{256} + \frac{1}{4} + r^2 \right) + 4 \times \frac{7}{16} \times \frac{1}{2} \times r.$$

This simplifies to

$$r^2 - r + \frac{1}{256} = 0.$$

So

$$r = \frac{1}{2} \pm \frac{3}{16}\sqrt{7}.$$

For each of these values of r, use the Spread law in $\overline{A_1 A_2 B}$

$$\frac{r}{25} = \frac{s}{Q}$$

and solve for Q, giving values

$$Q_1 = 1400 - 525\sqrt{7} \qquad \text{or} \qquad Q_2 = 1400 + 525\sqrt{7}.$$

To convert these answers back into distances, take square roots

$$d_1 = \sqrt{Q_1} \approx 3.3137\ldots \qquad \text{or} \qquad d_2 = \sqrt{Q_2} \approx 264.056\ldots.$$

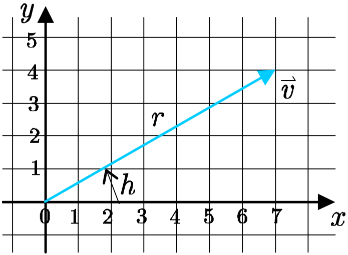
For more references on Rational Trigonometry:

Book: Divine Proportions: Rational Trigonometry to Universal Geometry (2005) Wild Egg Books

YouTube: user: **njwildberger**, Playlist: *WildTrig* (also of interest, Playlists: *MathFoundations, WildLinAlg, MathHistory, AlgTop, UnivHypGeom*)

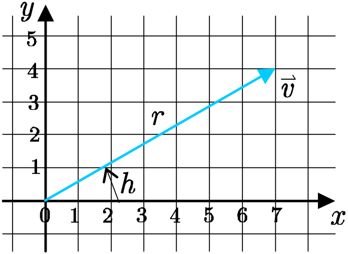
Papers: Various papers on the ArXiV by N J Wildberger.

Vector Trigonometry and Rotor coordinates



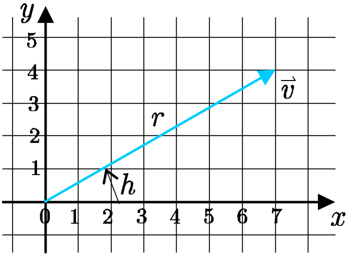
- $v = (x, y) = (7, 4)$ [Cartesian]

Vector Trigonometry and Rotor coordinates



- $v = (x, y) = (7, 4)$ [Cartesian]
- $v = (r, \theta) = \left(\sqrt{65},\ 0.519\,146\,114\,246\,523\ldots\right)$ [Polar]

Vector Trigonometry and Rotor coordinates



- $v = (x, y) = (7, 4)$ [Cartesian]
- $v = (r, \theta) = \left(\sqrt{65},\ 0.519\,146\,114\,246\,523\ldots\right)$ [Polar]
- $v = \left|r, h\right\rangle = \left|\sqrt{65},\ \frac{\sqrt{65}-7}{4}\right\rangle$ [Rotor] !!

Overview

- A) What are rotor coordinates?

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- B) Vector trigonometry

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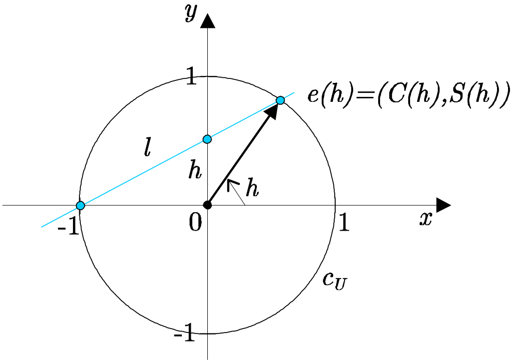
- C) Geometric application to quadrilaterals
- D) Kinematic application to Kepler-Newton orbits

A) What are rotor coordinates?

Rational parametrization of the unit circle



$$C\left(h\right) \equiv \frac{1-h^2}{1+h^2} \qquad \text{and} \qquad S\left(h\right) \equiv \frac{2h}{1+h^2}.$$

Definition

h is the **half-turn** of the unit vector $e\left(h\right)$.

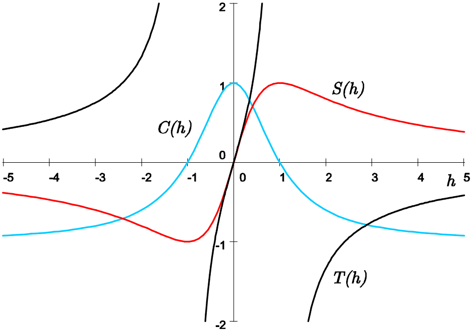
The rational circular functions

$$T(h) \equiv S(h)\,/\,C(h) \equiv \left(\frac{2h}{1+h^2}\right) / \left(\frac{1-h^2}{1+h^2}\right) = \frac{2h}{1-h^2}.$$

Lemma

$$C(h)^2 + S(h)^2 = 1$$

$$C(-h) = C(h) \qquad S(-h) = -S(h) \qquad \text{and} \qquad T(-h) = -T(h)\,.$$



Derivatives of the rational circular functions

$$C(h) \equiv \frac{1-h^2}{1+h^2} \qquad \text{and} \qquad S(h) \equiv \frac{2h}{1+h^2}.$$

Also define:

$$M(h) \equiv \frac{2}{1+h^2} = 1 + C(h) = \frac{S(h)}{h}$$

Lemma

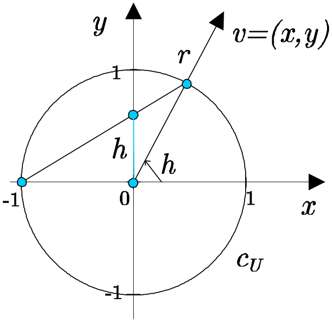
$$\frac{dC}{dh}(h) = -S(h)\ M(h) \qquad \textit{and} \qquad \frac{dS}{dh}(h) = C(h)\ M(h)$$

Lemma

Both $C(h)$ and $S(h)$ satisfy

$$\frac{1}{M(h)}\frac{d}{dh}\left(\frac{1}{M(h)}\frac{df}{dh}\right) + f = 0.$$

Rotor coordinates of a vector



- the **length** $r = r(\mathbf{v}) \equiv |\mathbf{v}| \equiv \sqrt{x^2 + y^2}$
- the **half-turn** $h = h(\mathbf{v}) = h(\mathbf{v}/|\mathbf{v}|)$

Definition

The numbers r and h are **rotor coordinates** for **v**. We write $\mathbf{v} = |\, r, h\rangle$.

The Half-turn formula

Theorem (Half-turn formula)

If $\mathbf{v} \equiv (x, y)$ *has length* $r \equiv \sqrt{x^2 + y^2}$ *and* $y \neq 0$*, then*

$$h(\mathbf{v}) = \frac{r - x}{y}.$$

Proof.

Use

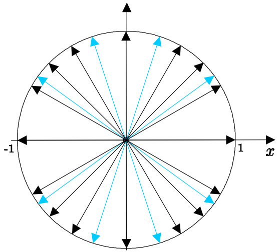
$$C(h) = \frac{x}{r} = \frac{1 - h^2}{1 + h^2} \qquad \textit{and} \qquad S(h) = \frac{y}{r} = \frac{2h}{1 + h^2}$$

to get

$$\frac{r - x}{y} = \frac{1 + h^2}{2h} - \frac{1 - h^2}{2h} = h. \blacksquare$$

$\square$

The Platonic directions



Some directions are far too familiar!

Example

$$30^\circ \approx 2-\sqrt{3} \qquad 45^\circ \approx \sqrt{2}-1 \qquad 60^\circ \approx 1/\sqrt{3} \qquad 90^\circ \approx 1$$

$$120^\circ \approx \sqrt{3} \qquad 135^\circ \approx \sqrt{2}+1 \qquad 150^\circ \approx 2+\sqrt{3} \qquad 180^\circ \approx \infty$$

Example

$$72^\circ \approx \sqrt{5-2\sqrt{5}} \qquad 144^\circ \approx \sqrt{5+2\sqrt{5}}$$

Other examples of half-turns

Here are some less familiar directions!

Example

If $\mathbf{v} \equiv (3,4)$ then $r = 5$ and $h = (5-3)/4 = 1/2$.

Example

If $\mathbf{v} \equiv (1,2)$ then $r = \sqrt{5}$ and

$$h = \frac{\sqrt{5}-1}{2} \approx 0.61803$$

Example

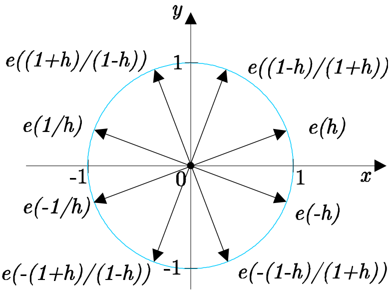
If $\mathbf{v} \equiv (-1,3)$ then $r = \sqrt{10}$ and

$$h = \frac{\sqrt{10}+1}{3} \approx 1.38743.$$

Theorem (Half-turn transformations)

Suppose that the vector $\mathbf{v}$ *has half-turn* h. *Then the reflection of* $\mathbf{v}$ *in the* x*-axis has half-turn* $-h$, *the reflection of* $\mathbf{v}$ *in the* y*-axis has half turn* $1/h$, *the vector* $-\mathbf{v}$ *has half-turn* $-1/h$, *while the reflection of* $\mathbf{v}$ *in the line* $y = x$ *and the rotation of* $\mathbf{v}$ *by a one-quarter of the full circle in the positive direction have respective half-turns*

$$\frac{1-h}{1+h} \quad \text{and} \quad \frac{1+h}{1-h}.$$



Rotations and the circle sum

Rotations can be described happily without angles:

$$\sigma_h \equiv \begin{pmatrix} C(h) & S(h) \\ -S(h) & C(h) \end{pmatrix} \qquad \text{and} \qquad \sigma_\infty \equiv \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Theorem (Circle sum)

For any half-turns h_1 and h_2,

$$\sigma_{h_1}\sigma_{h_2} = \sigma_h$$

where

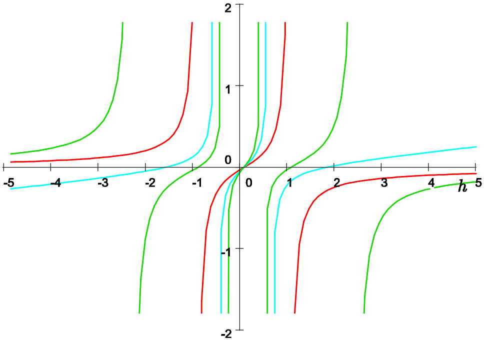
$$h = \frac{h_1 + h_2}{1 - h_1 h_2} \equiv h_1 \oplus h_2.$$

This defines the **circle sum** $h_1 \oplus h_2$ of half-turns. Associativity reduces to the algebraic identity

$$(h_1 \oplus h_2) \oplus h_3 = h_1 \oplus (h_2 \oplus h_3) = \frac{h_1 + h_2 + h_3 - h_1 h_2 h_3}{1 - (h_1 h_2 + h_2 h_3 + h_1 h_3)}.$$

Turn functions

$$\begin{aligned} h \oplus h &= \frac{2h}{1-h^2} \equiv U_2(h) \\ h \oplus h \oplus h &= \frac{3h-h^3}{1-3h^2} \equiv U_3(h) \\ h \oplus h \oplus h \oplus h &= \frac{4h-4h^3}{1-6h^2+h^4} \equiv U_4(h) \end{aligned}$$



Addition formulas

The functions C, S and T have addition formulas like $\cos\theta$, $\sin\theta$ and $\tan\theta$:

Theorem (C, S and T addition formulas)

$$
\begin{aligned}
C\left(h_1 \oplus h_2\right) &= C\left(h_1\right) C\left(h_2\right) - S\left(h_1\right) S\left(h_2\right) \\
S\left(h_1 \oplus h_2\right) &= C\left(h_1\right) S\left(h_2\right) + C\left(h_2\right) S\left(h_1\right)
\end{aligned}
$$

$$
T\left(h_1 \oplus h_2\right) = \frac{T\left(h_1\right) + T\left(h_2\right)}{1 - T\left(h_1\right) T\left(h_2\right)} = T\left(h_1\right) \oplus T\left(h_2\right)
$$

Proof.

These reduce to rational function identities: e.g.

$$
\frac{2\left(\frac{h_1+h_2}{1-h_1 h_2}\right)}{1-\left(\frac{h_1+h_2}{1-h_1 h_2}\right)^2} = \frac{\left(\frac{2h_1}{1-h_1^2}\right)+\left(\frac{2h_2}{1-h_2^2}\right)}{1-\left(\frac{2h_1}{1-h_1^2}\right)\left(\frac{2h_2}{1-h_2^2}\right)}.
$$

B) Vector trigonometry

Relative half-turns

The **(relative) half-turn between vectors** $\mathbf{v}_1 = \left| r_1, h_1 \right\rangle$ and $\mathbf{v}_2 = \left| r_2, h_2 \right\rangle$ is:

$$h = h\left(\mathbf{v}_1, \mathbf{v}_2\right) \equiv \frac{h_2 - h_1}{1 + h_1 h_2} = h_2 \oplus \left(-h_1\right).$$

It follows that

$$h_1 \oplus h = h_2.$$

The relative half-turn is an oriented quantity, $h\left(\mathbf{v}_2, \mathbf{v}_1\right) = -h\left(\mathbf{v}_1, \mathbf{v}_2\right)$.

Example

If $\mathbf{v}_1 \equiv (3, 2)$ and $\mathbf{v}_2 = (1, 5)$ then

$$h\left(\mathbf{v}_1, \mathbf{v}_2\right) = h_2 \oplus \left(-h_1\right) = \frac{\left(\frac{\sqrt{26}-1}{5}\right) - \left(\frac{\sqrt{13}-3}{2}\right)}{1 + \left(\frac{\sqrt{26}-1}{5}\right)\left(\frac{\sqrt{13}-3}{2}\right)} = \sqrt{2} - 1 \approx 45^\circ.$$

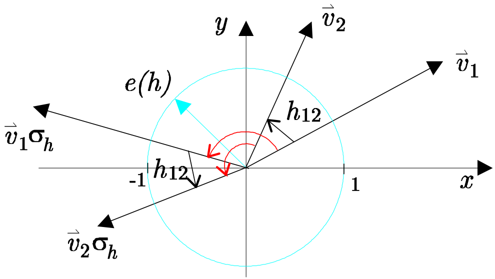
Invariance under rotation

The relative half-turn is invariant under rotations:

Theorem (Half-turn invariance)

For vectors $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *and any half turn* h

$$h(\mathbf{v}_1, \mathbf{v}_2) = h(\mathbf{v}_1\sigma_h, \mathbf{v}_2\sigma_h).$$



Relative half-turn formula

Theorem (Relative half-turn formula)

If $\mathbf{v}_1 \equiv (x_1, y_1)$ *and* $\mathbf{v}_2 \equiv (x_2, y_2)$ *with* $r_1 \equiv r(\mathbf{v}_1)$ *and* $r_2 \equiv r(\mathbf{v}_2)$, *then*

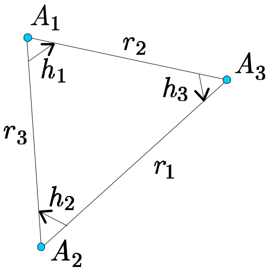
$$h = h(\mathbf{v}_1, \mathbf{v}_2) = \frac{y_1(r_2 - x_2) - y_2(r_1 - x_1)}{y_1 y_2 + (r_1 - x_1)(r_2 - x_2)}.$$

Example

If $\mathbf{v}_1 \equiv (3, 2)$ and $\mathbf{v}_2 = (1, 5)$ then

$$h(\mathbf{v}_1, \mathbf{v}_2) = \frac{2\left(\sqrt{26} - 1\right) - 5\left(\sqrt{13} - 3\right)}{2 \times 5 + \left(\sqrt{13} - 3\right)\left(\sqrt{26} - 1\right)} = \sqrt{2} - 1 \approx 45^\circ.$$

Cross law



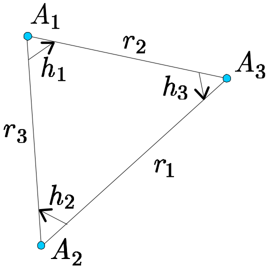
Theorem (Cross law-rotor form)

$$r_3^2 = r_1^2 + r_2^2 - 2r_1 r_2 C\left(h_3\right).$$

Corollary

$$h_3^2 = \frac{r_3^2 - \left(r_1 - r_2\right)^2}{\left(r_1 + r_2\right)^2 - r_3^2} = \frac{\left(r_1 - r_2 - r_3\right)\left(r_2 - r_1 - r_3\right)}{\left(r_1 + r_2 + r_3\right)\left(r_1 + r_2 - r_3\right)}.$$

Triangle half-turn formula



Theorem (Spread law-rotor form)

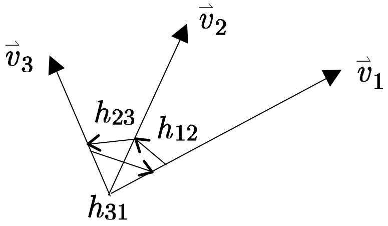
$$\frac{S(h_1)}{r_1} = \frac{S(h_2)}{r_2} = \frac{S(h_3)}{r_3}.$$

Theorem (Triangle half-turn formula)

$$h_1 h_2 + h_1 h_3 + h_2 h_3 = 1.$$

This replaces $\theta_1 + \theta_2 + \theta_3 = 3.14159265358979323846\,26\,434\ldots$

Three concurrent vectors



Theorem (Triple half-turn formula)

If $h_{12} \equiv h(\mathbf{v}_1, \mathbf{v}_2)$, $h_{23} \equiv h(\mathbf{v}_2, \mathbf{v}_3)$ *and* $h_{13} \equiv h(\mathbf{v}_1, \mathbf{v}_3)$*, then*

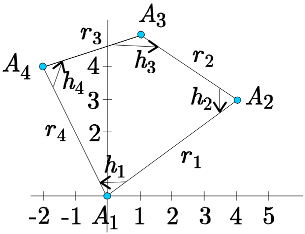
$$h_{12} + h_{23} + h_{31} = h_{12} h_{23} h_{31}.$$

Proof.

$$\frac{h_2 - h_1}{1 + h_1 h_2} + \frac{h_3 - h_2}{1 + h_2 h_3} + \frac{h_1 - h_3}{1 + h_3 h_1} = \left(\frac{h_2 - h_1}{1 + h_1 h_2}\right)\left(\frac{h_3 - h_2}{1 + h_2 h_3}\right)\left(\frac{h_1 - h_3}{1 + h_3 h_1}\right).$$

C) Geometric application to quadrilaterals

Quadrilateral half-turn formula



$$h_1 = \frac{5}{11}\sqrt{5} - \frac{2}{11}$$

$$h_2 = \frac{5}{17}\sqrt{13} - \frac{6}{17}$$

$$h_3 = \frac{1}{9}\sqrt{130} + \frac{7}{9}$$

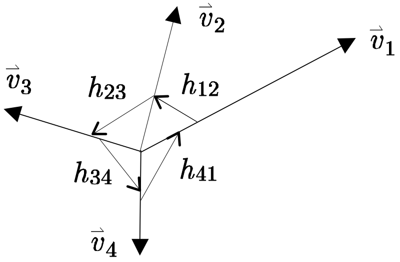
$$h_4 = \frac{1}{7}\sqrt{50} - \frac{1}{7}$$

Theorem (Quadrilateral half-turn formula)

If $h_1 \equiv h\left(\overrightarrow{A_1A_2}, \overrightarrow{A_1A_4}\right)$, $h_2 \equiv h\left(\overrightarrow{A_2A_3}, \overrightarrow{A_2A_1}\right)$, $h_3 \equiv h\left(\overrightarrow{A_3A_4}, \overrightarrow{A_3A_2}\right)$ *and* $h_4 \equiv h\left(\overrightarrow{A_4A_1}, \overrightarrow{A_4A_3}\right)$, *then*

$$h_1 + h_2 + h_3 + h_4 = h_1h_2h_3 + h_1h_2h_4 + h_1h_3h_4 + h_2h_3h_4.$$

Quadruple half-turn formula



Theorem (Quadruple half-turn formula)

If $h_{12} \equiv h(\mathbf{v}_1, \mathbf{v}_2)$, $h_{23} \equiv h(\mathbf{v}_2, \mathbf{v}_3)$, $h_{34} \equiv h(\mathbf{v}_3, \mathbf{v}_4)$ *and* $h_{41} \equiv h(\mathbf{v}_4, \mathbf{v}_1)$, *then*

$$h_{12} + h_{23} + h_{34} + h_{41} = h_{12}h_{23}h_{34} + h_{12}h_{23}h_{41} + h_{12}h_{31}h_{41} + h_{23}h_{34}h_{41}.$$

Triple quad formula

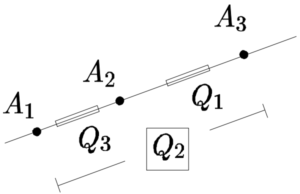
Quadrance: $Q([x_1, y_1], [x_2, y_2]) \equiv (x_2 - x_1)^2 + (y_2 - y_1)^2$

Theorem (Triple quad formula)

If $Q_1 \equiv Q(A_2, A_3)$, $Q_2 \equiv Q(A_1, A_3)$ *and* $Q_3 \equiv Q(A_1, A_2)$, *then*

$$(Q_1 + Q_2 + Q_3)^2 = 2\left(Q_1^2 + Q_2^2 + Q_3^2\right)$$

precisely when A_1, A_2 *and* A_3 *are collinear.*

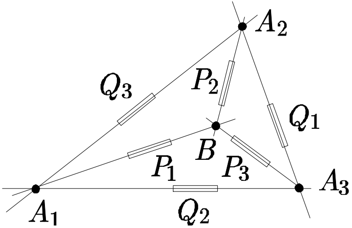


Quadrea: $\mathcal{A}\left(\overline{A_1 A_2 A_3}\right) \equiv (Q_1 + Q_2 + Q_3)^2 - 2\left(Q_1^2 + Q_2^2 + Q_3^2\right)$

Tartaglia's four-point relation

The Triple quad formula may be rewritten as

$$\begin{vmatrix} 2Q_1 & Q_1+Q_2-Q_3 \\ Q_1+Q_2-Q_3 & 2Q_2 \end{vmatrix} = 0.$$



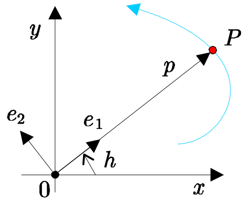
Theorem (Tartaglia's four-point relation)

$$\begin{vmatrix} 2P_1 & P_1+P_2-Q_3 & P_1+P_3-Q_2 \\ P_1+P_2-Q_3 & 2P_2 & P_2+P_3-Q_1 \\ P_1+P_3-Q_2 & P_2+P_3-Q_1 & 2P_3 \end{vmatrix} = 0.$$

D) Kinematic application to Newton-Kepler orbits

Kinematics in rotor coordinates

$$
\begin{aligned}
\mathbf{e}_1 &= (C(h), S(h)) \\
\mathbf{e}_2 &= (-S(h), C(h)) \\
\frac{d\mathbf{e}_1}{dt} &= M(h)\,\dot{h}\,\mathbf{e}_2 \\
\frac{d\mathbf{e}_2}{dt} &= -M(h)\,\dot{h}\,\mathbf{e}_1
\end{aligned}
$$



Theorem

If the position of P is $\mathbf{p} = r\,\mathbf{e}_1$ then

$$\mathbf{v} = \frac{d\mathbf{p}}{dt} = \dot{r}\,\mathbf{e}_1 + rM(h)\,\dot{h}\,\mathbf{e}_2$$

and

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \left(\ddot{r} - rM^2(h)\,\dot{h}^2\right)\mathbf{e}_1 + \frac{1}{r}\frac{d}{dt}\left(r^2 M(h)\,\dot{h}\right)\mathbf{e}_2.$$

Theorem (Conservation of angular momentum)

If $\mathbf{F} = m\,\mathbf{a}$ is central, i.e $\mathbf{F}(\mathbf{p}) \equiv -F(r(\mathbf{p}))$ is in the direction of $\mathbf{e}_1$, then

$$r^2 M(h)\,\dot{h} = c.$$

Also

$$-F(r) = m\left(\ddot{r} - rM^2(h)\,\dot{h}^2\right).$$

We wish to find $r = r(h)$. Set $w = w(h) \equiv 1/r$. Then

$$\begin{aligned}
\dot{r} &= -\frac{\dot{w}}{w^2} = -\frac{1}{w^2}\frac{dw}{dh}\dot{h} = -\frac{c}{M(h)}\frac{dw}{dh} \\
\ddot{r} &= \frac{c\dot{h}}{M^2(h)}\frac{dM}{dh}\frac{dw}{dh} - \frac{c}{M(h)}\dot{h}\frac{d^2w}{dh^2} = -c\dot{h}\frac{d}{dh}\left(\frac{1}{M(h)}\frac{dw}{dh}\right).
\end{aligned}$$

The differential equation

So

$$-\frac{F\left(1/w\right)}{m}=-\frac{c^{2}w^{2}}{M\left(h\right)}\frac{d}{dh}\left(\frac{1}{M\left(h\right)}\frac{dw}{dh}\right)-\frac{1}{w}M^{2}\left(h\right)\left(\frac{cw^{2}}{M\left(h\right)}\right)^{2}.$$

Theorem

For a central force field $F\left(r\right)$ and $w\equiv 1/r$,

$$\frac{1}{M\left(h\right)}\frac{d}{dh}\left(\frac{1}{M\left(h\right)}\frac{dw}{dh}\right)+w=\frac{F\left(1/w\right)}{c^{2}mw^{2}}.$$

Corollary

For an inverse-square central force, there is $k>0$ with

$$\frac{1}{M\left(h\right)}\frac{d}{dh}\left(\frac{1}{M\left(h\right)}\frac{dw}{dh}\right)+w=k.$$

Homogeneous case: $w\left(h\right)=C\left(h\right),S\left(h\right)$, Particular solution: $w=k$.

Kepler-Newton orbits

Theorem

For an inverse-square central force field, the rotor coordinates r and h of the motion satisfy

$$\frac{1}{r} = aC(h) + bS(h) + k$$

where a and b are constants that depend on initial conditions.

Use $C(h) = x/r$ and $S(h) = y/r$ to get

$$\frac{1 - ax - by}{r} = k$$

so that

$$(1 - ax - by)^2 = k^2 \left(x^2 + y^2\right).$$

This is a conic with focus $[0, 0]$, directrix $ax + by = 1$ and eccentricity e where $e^2 = \left(a^2 + b^2\right)/k^2$.
$e^2 > 1 \iff$ hyperbola, $e^2 = 1 \iff$ parabola, $e^2 < 1 \iff$ ellipse.

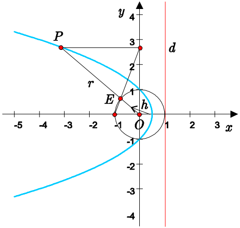
The parabolic case

Choose $a = k = 1$ and $b = 0$:

$$(1-x)^2 = x^2 + y^2$$

or

$$y^2 = 1 - 2x.$$



Conservation of angular momentum gives:

$$\frac{dh}{dt} = \frac{1}{r} = M(h) = \frac{2}{1+h^2}.$$

An easy integration:

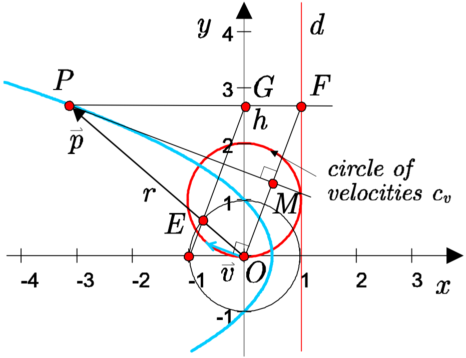
$$\int 1 + h^2 \; dh = h + \frac{h^3}{3} = \int 2 \; dt = 2t$$

Also we may derive:

$$x = \frac{1-h^2}{2} = 1 - r \qquad \text{and} \qquad y = h.$$

Circle of velocities

The motion relates naturally to the geometry of the parabola. In particular the circle of velocities is as shown.



Paper: Rotor Coordinates and Vector Trigonometry (N J Wildberger)
THANK YOU!